M. Math Exam

Question 1

For a fixed j, we need to find $P_{ij} = P(X_{n+1} = i | X_n = j)$ for all i. We have $P(X_n = j+1 | X_{n-1} = j) = p, P(X_n = 0 | X_{n-1} = j) = p$ and $P(X_n = k | X_{n-1} = j) = 0$ for all other values of k.

If T_0 denotes the time of return to zero starting from 0, we have that $T_0 = k$ occurs when we have all successes in the first $k-1$ tosses and the k^{th} toss is a failure. Therefore $P_0(T_0 = k) = p^{k-1}(1-p)$, for $k \geq 1$, which is a Geometric distribution with parameter p. This implies that $P_1(T_1 < \infty) = 1$. So by Proposition 6.3.5 of Athreya and Sunder 2008, we have that 0 is recurrent.

Also, from 0 it is possible to reach any state $k \geq 1$ with positive probability and therefore the Markov chain is irreducible. Thus all states are recurrent.

Question 2

(i) If $y = x$, then we have by Proposition 6.3.14 of Athreya and Sunder 2008 that x is transient and $\sum_n P_{xx}^{(n)} < \infty$. Here $P_{xx}^{(n)} = P_0(X_n = x)$. But we also have

$$\sum_n P_{xx}^{(n)} = \sum_n E_x(\mathbf{1}(X_n = x)) = E_x \sum_n \mathbf{1}(X_n = x) \tag{1}$$

and the final term is the expected number of visits to x starting from x. In the above $\mathbf{1}(.)$ refers to the indicator function. Using Fubini's theorem gives the necessary interchange.

If $y \neq x$, and suppose there is a positive probability of eventually reaching y from x. Starting from y, we then again have expected finite number of visits to y. Thus again $G(x, y) < \infty$ in this case.

Finally, if $y \neq x$, and suppose there is zero probability of reaching y from x, we have $G(x, y) = 0$ in this case.

(ii) As in (1), we have

$$G(x,y) = \sum_n P_{xy}^{(n)}.$$

Since $G(x,y) < \infty$, we have $\lim_n P_{xy}^{(n)} = 0$.

(iii) Let $x_1, ..., x_t$ be the states of the Markov chain and suppose they are all transient. We then have

$$\lim_n P_{x_1,x_i}^{(n)} = 0 \text{ for all } i. \tag{2}$$

But $\sum_{i=1}^t P_{x_1,x_i}^{(n)} = 1$ since starting from x_1, the Markov chain must be in one of the t states at time n. Allowing $n \to \infty$, we have $\sum_{i=1}^t \lim_n P_{x_1,x_i}^{(n)} = 1$ and here we interchange limits since the sum is finite. But thus contradicts (2).

Question 3

(i) We have $P_{ij} = P(X_{n+1} = j | X_n = i)$ and $P_{ij}^2 = \sum_k P_{ik} P_{kj}$. Also,

$$\begin{aligned}
&P(X_{2n+2} = j | X_{2n} = i) \\
&= \sum_k P(X_{2n+2} = j | X_{2n+1} = k, X_{2n} = i) P(X_{2n+1} = k | X_{2n} = i) \\
&= \sum_k P(X_{2n+2} = j | X_{2n+1} = k) P(X_{2n+1} = k | X_{2n} = i) \qquad (3) \\
&= \sum_k P_{ik} P_{kj}
\end{aligned}$$

where (3) is due to the Markovian nature. Thus P_{ij}^2 is the transition probability matrix for $\{X_{2n}\}_n$.

(ii) Let t be the number of states. Since π is a stationary distribution, we have

$$\sum_{i=1}^t \pi(i) P_{ij} = \pi(j) \text{ for all } i, j. \tag{4}$$

Using this we have

$$\begin{aligned}\sum_{i=1}^{t}\pi(i)P_{ij}^2 &= \sum_{i=1}^{t}\pi(i)\sum_{k=1}^{t}P_{ik}P_{kj}\\ &= \sum_{k=1}^{t}\sum_{i=1}^{t}\pi(i)P_{ik}P_{kj}\\ &= \sum_{k=1}^{t}\pi(k)P_{kj}=\pi(j)\end{aligned}$$

Thus π is also a stationary distribution for P^2.

(iii) No. Take state space to be $\{0,1\}$, $P(X_{n+1}=1|X_n=0)=1$ and $P(X_{n+1}=0|X_n=1)=1$. Starting from 0, the Markov chain oscillates between 0 and 1 at alternate times. Here $\pi=(1,0)$ is a stationary distribution for P^2 but not for P.

Question 4

(i) We condition on the first step X_1, to get

$$P_x(T_y=n+1)=\sum_{z\neq y}P_x(T_y=n+1|X_1=z)P_x(X_1=z)=\sum_{z\neq y}P_z(T_y=n)P_{xz} \tag{5}$$

For the second equality we argue this way: if in the first step we reach $z\neq y$, then n time units are needed to reach y from z. From Markov property the equation then follows.

(ii) We have

$$\rho_{xy}=P_x(T_y<\infty)=P_x(T_y=1)+\sum_{k\geq 2}P_x(T_y=k)=P_{xy}+\sum_{k\geq 2}P_x(T_y=k).$$

For the second term we proceed as follows. Summing (5) over $n\geq 1$, we have

$$\sum_{n\geq 1}P_x(T_y=n+1)=\sum_{z\neq y}\sum_{n\geq 1}P_z(T_y=n)P_{xz}=\sum_{z\neq y}P_{xz}\rho_{zy}$$

since the inner summation in the middle term is precisely the probability of eventually reaching y from z. In the above, the iterated sums are not necessarily finite and therefore not directly interchangeable. As before, using Fubini's theorem gives the necessary interchange.